MATH 2028 Honours Advanced Calculus II
2024-25 Term 1
Problem Set 1
due on Sep 20, 2024 (Friday) at 11:59PM

Instructions: You are allowed to discuss with your classmates or seek help from the TAs but you are required to write/type up your own solutions. You can either type up your assignment or scan a copy of your written assignment into ONE PDF file and submit through CUHK Blackboard on/before the due date. Please remember to write down your name and student ID. **No late homework will be accepted.**

Notations: Throughout this problem set, we use R to denote a rectangle in $\mathbb{R}^n$, and $B_\delta(p) \subset \mathbb{R}^n$ to denote the open ball of radius δ centered at p.

Problems to hand in

1. Let $f : R = [0,1] \times [0,1] \to \mathbb{R}$ be a bounded function defined by
$$f(x,y) := \begin{cases} 1 & \text{if } y < x, \\ 0 & \text{if } y \geq x. \end{cases}$$
Prove, using the definition, that f is integrable and find $\int_R f \, dV$.

2. Let $f : R = [0,1] \times [0,1] \to \mathbb{R}$ be the function
$$f(x,y) = \begin{cases} 1/q & \text{if } x, y \in \mathbb{Q} \text{ and } y = p/q \text{ where } p, q \in \mathbb{N} \text{ are coprime}, \\ 0 & \text{otherwise.} \end{cases}$$
Prove, using the definition, that f is integrable and find $\int_R f \, dV$.

3. Suppose $f : R \to \mathbb{R}$ is a non-negative *continuous* function such that $f(p) > 0$ at some $p \in R$. Prove that $\int_R f \, dV > 0$.

4. Let $f : R \to \mathbb{R}$ be a bounded integrable function. Prove that $|f|$ is also integrable on R and $\left| \int_R f \, dV \right| \leq \int_R |f| \, dV$.

5. Let $f : R \to \mathbb{R}$ be a bounded integrable function. Suppose p is an interior point of R at which f is continuous. Prove that
$$\lim_{\delta \to 0^+} \frac{1}{\text{Vol}(B_\delta(p))} \int_{B_\delta(p)} f \, dV = f(p).$$

Suggested Exercises

1. Let $f, g : R \to \mathbb{R}$ be bounded integrable functions. Prove that $f + g$ is integrable on R and
$$\int_R (f+g) \, dV = \int_R f \, dV + \int_R g \, dV.$$

2. Let $f : R \to \mathbb{R}$ be a bounded integrable function defined on a rectangle $R \subset \mathbb{R}^n$. Suppose $g : R \to \mathbb{R}$ is a bounded function such that $g(x) = f(x)$ except for finitely many $x \in R$. Show that g is integrable and $\int_R g\, dV = \int_R f\, dV$.

Challenging Exercises

1. Let f be a bounded integrable function on R. Prove that for any $\epsilon > 0$, there exists some $\delta > 0$ such that whenever $\mathcal{P}$ is a partition of R with $\operatorname{diam}(Q) < \delta$ for all $Q \in \mathcal{P}$, and $x_Q \in Q$ is any arbitrarily chosen point inside $Q \in \mathcal{P}$, we have

$$\left| \sum_{Q \in \mathcal{P}} f(x_Q)\operatorname{Vol}(Q) - \int_R f\, dV \right| < \epsilon.$$

(The sum in the above expression is what we usually call the "Riemann sum"!)